Homeomorphism and Topological Properties

Spaces that aren't $\mathbb{R}$eal

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If two spaces have different topological properties, then they aren't homeomorphic.

Let's give an example:

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$[0, \infty)$ is homeomorphic to $Y := \mathbb{N} \times [0, 1)$ in dictionary order
Namely, our homeomorphism is $f : Y \mapsto \mathbb{R} : (x, y) \mapsto x + y$

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We've tried $A := \mathbb{N} \times [0, 1)$ in dictionary order.
Now let's try the opposite. $B := [0, 1) \times \mathbb{N}$ in dictionary order.
I believe we referred to this as "blowing up the $\mathbb{R}$"

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I call it an infinitesimal harp.
If we remove a point from $\mathbb{R}$, it's no longer path connected. By contrast, for a normal unit square, $[0,1] \times [0,1]$, if we were to remove a point, we would still be path connected.

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A space is contractible if it is deformable to a point in finite time.

$\mathbb{R}$ is contractible , as $\forall n \in \mathbb{N}$ (n,n+1) is homeomorphic to $(0, \frac{1}{2^{n+1}})$, by the map $f(x) = \frac{x-n}{2^{n+1}}$, which is homeomorphic to $(\frac{2^n}{2^{n+1}}, \frac{1+2^n}{2^{n+1}})$

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Now that we've gotten to [0,1), we can divide $\forall n \in \mathbb{N}$
But what if for $\mathbb{N} \times [0,1)$ we used a bigger set than $\mathbb{N}$?
What if we used ω_1?
Our ray would be a lot longer.

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So what are we to learn?
$\mathbb{R}$ is a little special.

- It's Hausdorff, but not discrete.
- It's path-connected, but barely.
- And it's infinitely long, but not too long.